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Intrinsic Magnetic Flux of the Electron's Orbital and Spin Motion

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In analogy with the fact that there are magnetic moments associated respectively with the electron's orbital and spin motion in an atom we present several analyses on a proposal to introduce a concept of intrinsic magnetic flux associated with the electron's orbital and spin motion. It would be interesting to test or to demonstrate Faraday's and Lenz's laws of electromagnetic induction arising directly from the flux change due to transition of states in an atom and to examine applications of this concept of intrinsic flux.

KEY WORDS: electron intrinsic orbital and spin magnetic flux.

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1. INTRODUCTION

Using the semi-classical Bohr-Sommerfeld quantization rule (Landau and Lifshitz, 2002; Merzbacher, 1998) Kittel presented an analysis showing that an electron in circular motion in a constant external magnetic field would enclose a magnetic flux which admits only a discrete set of values (Kittel, 1996; Saglam and Boyacioglu, 2002a, 2002b):

$$\Phi_k = \left(k + \frac{1}{2}\right)\frac{h}{e}, \quad k = 0, 1, 2, \ldots, \tag{1}$$

where $e > 0$ is the elementary charge and h is the Planck's constant. One can arrive at this result in a formal manner, using quantum mechanical results on the motion of a particle of charge q and mass m_q in an external magnetic field. A constant magnetic field along the z-axis of magnitude B is associated with a vector

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potential

$$\boldsymbol{A} = (A_x, A_y, A_z), \quad A_x = -\frac{1}{2}By, \ A_y = \frac{1}{2}Bx, \ A_z = 0. \tag{2}$$

The magnitude of $\boldsymbol{A}$ is $A = \frac{1}{2}Br$, where $r = \sqrt{x^2 + y^2}$. A charged classical particle will spiral along the z-axis, i.e., its behaviour can be decomposed into a uniform motion along the z-axis and a circular motion on the x–y plane of angular frequency

$$\omega_q = qB/m_q. \tag{3}$$

When quantized we will find that in the Heisenberg picture the equations of motion of the particle resemble the corresponding classical Hamilton's equations of motion, and consequently the particle's motion also resembles that of its classical counter part (Capri, 1985), i.e., we can identify a circular motion on the x–y plane which enables us to define a magnetic flux enclosed by the orbit of the circular motion. A brief analysis is set out in the next two paragraphs.

First we observe that the Hamiltonian of the particle can be decomposed into a sum of two commuting parts, i.e.,

$$\widehat{H} = \frac{1}{2m_q}(\widehat{\boldsymbol{p}} - q\boldsymbol{A})^2 = \widehat{H}_z + \widehat{H}_{xy} \tag{4}$$

where

$$\widehat{H}_z = \frac{1}{2m_q}\widehat{p}_z^2, \tag{5}$$

and

$$\widehat{H}_{xy} = \frac{1}{2m_q}\left\{(\widehat{p}_x - qA_x)^2 + (\widehat{p}_y - qA_y)^2\right\} \tag{6}$$

$$= \frac{1}{2m_q}\left(\widehat{p}_x^2 + \widehat{p}_y^2\right) + \frac{q^2B^2}{8m_q}\left(\widehat{x}^2 + \widehat{y}^2\right) - \frac{qB}{2m_q}(\widehat{x}\widehat{p}_y - \widehat{y}\widehat{p}_x). \tag{7}$$

Introduce two new operators $\widehat{v}_x$ and $\widehat{v}_y$ by

$$\widehat{v}_x = \frac{1}{m_q}(\widehat{p}_x - qA_x), \quad \widehat{v}_y = \frac{1}{m_q}(\widehat{p}_y - qA_y). \tag{8}$$

Then we have

$$\widehat{H}_{xy} = \frac{1}{2}m_q\left(\widehat{v}_x^2 + \widehat{v}_y^2\right). \tag{9}$$

In the Heisenberg picture the Heisenberg equations of motion for $\widehat{x}$ and $\widehat{y}$ are governed by $\widehat{H}_{xy}$, since $\widehat{x}$ and $\widehat{y}$ commute with $\widehat{H}_z$. We get:

1. Relations between $\widehat{x}$ and $\widehat{v}_x$, and between $\widehat{y}$ and $\widehat{v}_y$:

$$\frac{d\widehat{x}(t)}{dt} = \widehat{v}_x(t), \quad \frac{d\widehat{y}(t)}{dt} = \widehat{v}_y(t). \tag{10}$$

2. Equations of motion for $\widehat{x}$ and $\widehat{y}$:

$$\frac{d\widehat{v}_x(t)}{dt} = \omega_q \frac{d\widehat{y}(t)}{dt}, \quad \frac{d\widehat{v}_y(t)}{dt} = -\omega_q \frac{d\widehat{x}(t)}{dt}, \tag{11}$$

   where $\omega_q = qB/m_q$. Integrating once gives

$$\widehat{v}_x(t) - \widehat{v}_x(0) = \omega_q(\widehat{y}(t) - \widehat{y}(0)), \quad \widehat{v}_y(t) - \widehat{v}_y(0) = -\omega_q(\widehat{x}(t) - \widehat{x}(0)). \tag{12}$$

   These equations can be rewritten as

$$\widehat{v}_x(t) = \omega_q \widehat{y}(t) - \omega_q \widehat{y}_0, \quad \widehat{v}_y(t) = -\omega_q \widehat{x}(t) + \omega_q \widehat{x}_0, \tag{13}$$

   where

$$\widehat{x}_0 = \widehat{x}(0) + \frac{1}{\omega_q}\widehat{v}_y(0), \quad \widehat{y}_0 = \widehat{y}(0) - \frac{1}{\omega_q}\widehat{v}_x(0). \tag{14}$$

Now, introduce an operator $\widehat{r}^2(t)$ defined by

$$\widehat{r}^2(t) = (\widehat{x}(t) - \widehat{x}_0)^2 + (\widehat{y}(t) - \widehat{y}_0)^2. \tag{15}$$

Then we have

$$\widehat{r}^2(t) = \frac{1}{\omega_q^2}(\widehat{v}_x^2(t) + \widehat{v}_y^2(t)) = \frac{2}{m_q \omega_q^2}\widehat{H}_{xy}, \tag{16}$$

which renders $\widehat{r}^2(t)$ time independent in the Heisenberg picture. It follows that we have

$$\widehat{v}_x^2(t) + \widehat{v}_y^2(t) = (\omega_q \widehat{r}(t))^2 = (\omega_q \widehat{r}(0))^2 = \widehat{v}_x^2(0) + \widehat{v}_y^2(0). \tag{17}$$

Denoting $\widehat{r}^2(0)$ simply by $\widehat{r}^2$ we get

$$\widehat{r}^2 = \frac{2}{m_q \omega_q^2}\widehat{H}_{xy} \quad \text{or} \quad \frac{1}{2} m_q \omega_q^2 \widehat{r}^2 = \widehat{H}_{xy}. \tag{18}$$

Classically the charged particle's circular motion in the x–y plane about an origin (x_0, y_0) is fixed by the initial conditions. A circular motion of radius r and angular frequency ω_q about an origin (x_0, y_0) in the x–y plane is explicitly describable by

$$x(t) = r\sin(\omega_q t + \lambda) + x_0, \quad y(t) = r\cos(\omega_q t + \lambda) + y_0. \tag{19}$$

Note that (x_0, y_0) is the position of the origin of the circular motion. The position of the particle at $t = 0$ is $(x(0), y(0))$ which is different from (x_0, y_0). The radius

r of the circular motion clearly satisfies

$$r^2 = (x(t) - x_0)^2 + (y(t) - y_0)^2. \tag{20}$$

Let $v_x(t) = dx(t)/dt$ and $v_y(t) = dy(t)/dt$. Using Eq. (19) we can verify the following relations:

$$x_0 = x(0) + \frac{1}{\omega_q} v_y(0), \quad y_0 = y(0) - \frac{1}{\omega_q} v_x(0), \tag{21}$$

and

$$v_x^2(t) + v_y^2(t) = \omega_x^2 r^2. \tag{22}$$

The particle's energy E_{xy} for the circular motion can be directly related to r:

$$E_{xy} = \frac{1}{2} m_q v^2 = \frac{1}{2} m_q \omega_q^2 r^2 \quad \text{or} \quad r^2 = \frac{2}{m_q \omega_q^2} E_{xy}. \tag{23}$$

By comparing with the quantum results presented earlier we can interpret $\widehat{r}$ as the radius operator for the circular motion in the x–y plane about the origin $\widehat{x}_0$ and $\widehat{y}_0$ in the sense that the quantized numerical values of the square of the radius are identified with the eigenvalues of $\widehat{r}^2$ (Capri, 1985). We can interpret this result as the qunatization of the circular orbits, as in the Bohr model of the atom. The eigenvalues R_n^2 of $\widehat{r}^2$ can be written down, i.e.,

$$R_n^2 = \frac{2\hbar}{m_q \omega_q} (n + 1/2). \tag{24}$$

These values are obtained in terms of the eigenvalues of the Hamiltonian $\widehat{H}_{xy}$ which are known to be $(n + 1/2)\hbar\omega_q$ for some integer n (Capri, 1985). A knowledge of R_n^2 enables us to fix a corresponding area enclosed by the circular path. This in turn enables us to define the magnetic flux enclosed by a quantized orbit, i.e.,

$$\Phi_n = \pi R_n^2 B = \left(n + \frac{1}{2}\right) \frac{h}{q}, \tag{25}$$

which agrees with the result in Eq. (1).

One may wonder how this can be reconciled with the fact that generally quantum mechanics does not allow us to trace the path of a particle. In other words we cannot have a circular path fixed in space for the motion of a quantum particle in an external magnetic field. The answer is that we define the enclosed area in terms of a quantized value R_n^2 of the radius square without requiring a circular orbit rigidly fixed in space. In fact we do not have a rigid circular orbit for the particle to trace out because of the lack of a fixed origin. To fix the origin we need simultaneous values of $\widehat{x}_0$ and $\widehat{y}_0$; we cannot have these simultaneous values since $\widehat{x}_0$ and $\widehat{y}_0$ do not commute (Capri, 1985). Despite this, a knowledge of R_n^2

does enable us to have a well-defined numerical value of the area enclosed by an orbit without having a rigidly fixed orbit.

One may augue that a spiralling trajectory in three-dimensions with a non-zero velocity component v_z does not enclose an area in the x–y plane. However, we can choose a new coordinate system (x', y', z') with $x' = x$, $y' = y$, $z' = z - v_z t$, i.e., a coordinate system moving along z-axis with velocity v_z. Then the particle in the x-y plane at $t = 0$ will be seen to move round a circle lying in the $x' - y'$ plane at a later time in Kittel's presentation. This enables us to establish a flux enclosed by the circle on the $x' - y'$ plane in the ususal sense.

It is well-known in the study of diamagnetism that a circulating charged particle gives rise to a circulating current I which in turn will give rise to a magnetic field, due to Faraday's law. The current will then enclose a magnetic flux (Kip, 1981). Our task is to formulate a quantum theory of induced magnetic flux due to the circular motion of a quantum particle. Note that such an induced flux is not necessarily related to the presence of an external magnetic field. In the example presented earlier we are concerned with the external magnetic flux enclosed by an orbit; we have not considered the magnetic flux generated by the current due to the orbital motion of the particle.

In this paper we shall present a study of magnetic flux generated by an electron in orbital motion and by its spin independent of an external magnetic field. In other words we aim to establish an intrinsic magnetic flux due to an electron's orbital motion and due to its spin. We shall show that different formulations lead to the same results.

Consider the simple case of an electron in a hydrogen atom. The 3-dimensional Euclidean space $\mathbb{R}^3$ can be parameterized either by rectangular Cartesian coordinates (x, y, z) or by the corresponding spherical coordinates (r, θ, φ). A normalised energy eigenfunction is of the form $\psi_{n\ell m}(r, \theta, \varphi) = R_{n\ell}(r) Y_{\ell m}(\theta, \varphi)$, where $R_{n\ell}(r)$ is the radial wave function, $Y_{\ell m}(\theta, \varphi)$ is a spherical harmonic, and n, ℓ, m are the usual quantum numbers, in particular m is the magnetic quantum number. The z-component angular momentum operator $\widehat{L}_z = -i\hbar\partial/\partial\varphi$ admits $\psi_{n\ell m}(r, \theta, \varphi)$ as an eigenfunction corresponding to the eigenvalue $L_z = m\hbar$. An energy eigenfunction $\psi_{n\ell m}(r, \theta, \varphi)$ with $m \neq 0$ describes an electron having a probability density $|\psi_{n\ell m}(r, \theta, \varphi)|^2$ of being at position (r, θ, φ) and executing a circular motion of radius (Spiegel, 1974) $a = r \sin\theta$ about z-axis. This is born out by the fact that due to the radial wave function and the θ-dependent part of the spherical harmonic being real-valued the probability current density arising from an energy eigenfunction $\psi_{n\ell m}(r, \theta, \varphi)$ is zero along the radial and the θ directions. There is a non-zero probability current density for the circular motion along the φ direction given by (Merzbacher, 1998; Cohen-Tannoudji *et al.*, 1977)

$$j(r, \theta, \varphi) = -\frac{i\hbar}{2m_e}\left(\psi^*_{n\ell m}\frac{\partial\psi^*_{n\ell m}}{a\partial\varphi} - \frac{\partial\psi_{n\ell m}}{a\partial\varphi}\psi_{n\ell m}\right) \tag{26}$$

$$= \frac{L_z}{m_e a} \, |\psi_{n\ell m}(r, \theta, \varphi)|^2 \, , \tag{27}$$

where m_e is the mass of the electron. The probability current across an elementary surface area (Spiegel, 1974) $rdrd\theta$ perpendicular to the flow is

$$j(r, \theta, \varphi) \, r dr \, d\theta = \frac{L_z}{m_e a} \, |\psi_{n\ell m}(r, \theta, \varphi)|^2 \, r dr \, d\theta. \tag{28}$$

A probability current density $j(r, \theta, \varphi)$ gives rise to an electric current density (Merzbacher, 1998; Cohen-Tannoudji *et al.*, 1977)

$$j_e(r, \theta, \varphi) = -ej(r, \theta, \varphi). \tag{29}$$

Here the negative sign shows that the electric current flows in an opposite direction to that of the probability current due to the negative nature of the electron charge. It follows that there is an electric current circling the z-axis in a circle of radius $a = r \sin\theta$ flowing perpendicularly across an elementary surface area $rdrd\theta$ given by

$$j_e(r, \theta, \varphi) \, r dr \, d\theta = -\frac{eL_z}{m_e a} \, |\psi_{n\ell m}(r, \theta, \varphi)|^2 \, r dr \, d\theta. \tag{30}$$

As a precursor let us derive the magnetic moment of an electron in orbital motion in an atom, a quantity well-known in quantum mechanics.

2. ORBITAL MAGNETIC MOMENT

We know from classical magnetostatics that an electrical current of magnitude I going round a circular loop of radius a gives rise to a magnetic moment of magnitude (Jackson, 1999) $M = \pi a^2 I$. A particle of charge q and mass m_q moving with speed v round a circle of radius a in the x-y plane will revolve with frequency $v/2\pi a$. This motion gives rise to a current I and a magnetic moment M_z along the z-axis given by

$$I = q \frac{v}{2\pi a} = \rho_q v \quad \text{and} \quad M_z = \frac{1}{2} q a v, \tag{31}$$

where $\rho_q = q/2\pi a$ may be interpreted as the effective charge density round the circle. This is related to its (classical) angular momentum $L_{cl} = m_q v a$ along the z-axis by (Jackson, 1999)

$$M_z = \frac{q}{2m_q} L_{cl}. \tag{32}$$

The quantum magnetic moment of an orbiting electron in an atom can be similarly worked out (Greiner, 1989). An orbiting electron gives rise to an electric current circulating the z-axis with current density given by Eq. (29). This results in a current in Eq. (30) flowing across surface element $rdr\,d\theta$. Such a current element $j_e(r, \theta\varphi)\, rdr\, d\theta$ in Eq. (30) circulating the z-axis in a circle of radius a gives rise to a magnetic moment element $d\mu_z^{(o)}$ given by

$$d\mu_z^{(o)} = \pi a^2 j_e(r, \theta, \varphi)\, rdr\, d\theta \tag{33}$$

$$= -\pi a^2 \frac{eL_z}{m_e a} |\psi_{n\ell m}(r, \theta, \varphi|^2\, rdr\, d\theta \tag{34}$$

$$= -\frac{\pi e L_z}{m_e} |\psi_{n\ell m}(r, \theta, \varphi)|^2\, r^2 \sin\theta\, dr\, d\theta, \quad \text{using } a = r\sin\theta. \tag{35}$$

The total magnetic moment is then

$$\mu_z^{(o)} = \int_0^\infty \int_0^\pi -\frac{\pi e L_z}{m_e} |\psi_{n\ell m}(r, \theta, \varphi)|^2\, r^2 \sin\theta\, dr\, d\theta \tag{36}$$

$$= -\frac{e}{2m_e} L_z. \tag{37}$$

We have used the normalised nature of $\psi_{n\ell m}(r, \theta, \varphi)$. This is the well-known result for the magnetic moment of a circulating electron in an atom.

3. ORBITAL MAGNETIC FLUX

3.1. The Problem

Our next objective is to establish a magnetic flux associated with the orbiting electron in a hydrogen atom. The idea is based on the familiar notion that an electric current flowing round a circle of radius a about the z-axis would enclose a magnetic flux generated by the current. This flux should be proportional to the current, the proportionality constant being the self-inductance (Lorrain and Corson, 1970). So, a current element in Eq. (30) circulating about the z-axis in a circle of radius a should enclose a magnetic flux element $d\Phi_z^{(o)}(r, \theta, \varphi)$ proportional to the current $j_e(r, \theta, \varphi)\, rdr\, d\theta$, i.e., we have

$$d\Phi_z^{(o)}(r, \theta, \varphi) = \mathcal{L}_e(a) j_e(r, \theta, \varphi)\, rdr\, d\theta, \tag{38}$$

where $\mathcal{L}_e(a)$ is the self-inductance. It is not clear what expression we should use for the self-inductance. One way to proceed is to seek a precedent, i.e., to see if there is any expression for a self-inductance of a circulating quantum current which would give the correct result for the enclosed magnetic flux. Fortunately there is indeed a precedent in the study of superconductivity. A supercurrent in a superconductor is due to the flow of electron pairs known as Cooper pairs (Feynman *et al.*, 1965). Each Cooper pair has a mass $m_c = 2m_e$ and a charge $q_c = -2e$ associated with it. When considering the magnetic flux enclosed by a supercurrent in a superconducting ring Wan and Harrison (Wan and Harrison,

1993) propose a model in terms of a current flowing round a circle of radius a with a self-inductance which is proportional to a Cooper pair's mass m_c and inversely proportional to the charge density $\rho_c = q_c/2\pi a$, i.e.,

$$\mathcal{L}_c(a) = \frac{m_c}{\rho_c^2} = m_c \left(\frac{2\pi a}{q_c}\right)^2. \tag{39}$$

Such a model leads to the correct quantized magnetic flux values enclosed by a superconducting ring. The theory has been generalized to apply to a number of superconducting circuits (Wan and Harrison, 1993; Wan and Fountain, 1996; Harrison and Wan, 1997; Wan and Fountain, 1998; Trueman and Wan, 2000; Wan, 2006). We shall assume a corresponding expression in Eq. (39) for our circulating current due to the motion of an electron, as opposed to a Cooper pair, i.e., we shall assume

$$\mathcal{L}_e(a) = \frac{m_e}{\rho_e^2} = m_e \left(\frac{2\pi a}{e}\right)^2. \tag{40}$$

We can gain an appreciation of such an expression for the self-inductance in terms of classical circuit theory. First consider an ideal resistanceless coil of radius r and self inductance $\mathcal{L}$ lying in the x-y-plane. Apply an external magnetic field of magnitude $B(t)$, where $B(t)$ increases from an initial zero value, and directed perpendicular along the positive z-direction to the coil. An *e.m.f.*, and hence an electric field $E(t)$ and a current $I(t)$, will be induced in the coil due to the change of the external magnetic flux $\Phi_{\text{ex}}(t)$ enclosed by the coil. The electric field will be tangentially directed, i.e.,

$$E(t) = E(t) i_\theta, \tag{41}$$

where i_θ is the unit vector along the polar angle θ, and $\boldsymbol{E}(t) > 0$ if $\boldsymbol{E}(t)$ is anti-clockwise, and $E(t) < 0$ if $\boldsymbol{E}(t)$ is clockwise. Similarly the current $I(t)$ is positive in the anti-clockwise direction and negative in the clockwise direction,

Faraday's and Lenz's laws imply a link between $\boldsymbol{E}(t)$ and $\Phi_{\text{ex}}(t)$:

$$e.m.f. = \oint \boldsymbol{E}(t) \cdot d\ell = -\frac{d\Phi_{\text{ex}}}{dt} \Rightarrow 2\pi r E(t) = -\frac{d\Phi_{\text{ex}}}{dt}. \tag{42}$$

Moreover, there will be an induced flux Φ_{ind} due to the induced current I, i.e., we have

$$\Phi_{\text{ind}} = \mathcal{L}_q(r) I(t), \tag{43}$$

where $\mathcal{L}_q(r)$ is the self-inductance. The circuit equation for the current flow in the coil is

$$-\frac{d\Phi_{\text{ex}}}{dt} - \frac{d\Phi_{\text{ind}}}{dt} = I(t)R = 0, \tag{44}$$

on account of zero resistance, i.e., $R = 0$. It follows that the total magnetic flux threading through the coil

$$\Phi_T = \Phi_{\text{ex}} + \Phi_{\text{ind}} \tag{45}$$

is a constant, a well-known result (Rose-Innes and Rhoderick, 1978). Moreover, we have, on account of Eq. (42),

$$2\pi r E(t) = \frac{d\Phi_{\text{ind}}}{dt} = \mathcal{L}_q(r)\frac{dI}{dt}. \tag{46}$$

Let us consider a particle of charge q and mass m_q constrained to move round a circle of radius r with tangential velocity v. Such a circular motion of the particle gives rise to a current $I = \rho_q v$. Now, suppose the particle's motion round the circle, and hence the current, is caused, by the *e.m.f.* induced by an applied flux $\Phi_{\text{ex}}(t)$. This means that the particle is accelerated from rest by an induced electric field $E(t)$ given by Eq. (42). We have (Kip, 1981)

$$m_q \frac{dv}{dt} = qE(t). \tag{47}$$

From Eqs. (46) and (47) we get

$$m_q \frac{dv}{dt} = q\frac{\mathcal{L}_q(r)}{2\pi r}\frac{dI}{dt} = \mathcal{L}_q(r)\left(\frac{q}{2\pi r}\right)\rho_q \frac{dv}{dt} \tag{48}$$

$$\Rightarrow \mathcal{L}_q(r) = m_q\left(\frac{2\pi r}{q}\right)^2 = \frac{m_q}{\rho_q^2}, \tag{49}$$

agreeing with the result in Eqs. (39) and (40). The resulting flux due to the current I has the value

$$\Phi_I = \mathcal{L}_q(r)I = \frac{2\pi}{q}L_{cl}. \tag{50}$$

3.2. Intrinsic Magnetic Flux of an Orbiting Electron

Returning to our main line of argument, we can obtain the magnetic flux due to an electron's orbital motion as follows:

1. First write down the expression in Eq. (38) for $d\Phi_z^{(o)}(r, \theta, \varphi)$ explicitly, i.e.,

$$d\Phi_z^{(o)}(r, \theta, \varphi) = \mathcal{L}_e(a) j_e(r, \theta, \varphi)\, r dr\, d\theta \tag{51}$$

$$= m_e\left(\frac{2\pi a}{e}\right)^2\left(-\frac{eL_z}{m_e a}\right)|\psi_{n\ell m}(r, \theta, \varphi)|^2\, r dr\, d\theta \tag{52}$$

$$= -\frac{(2\pi)^2 L_z}{e} \, |\psi_{n\ell m}(r, \theta, \varphi)|^2 \, r^2 \sin\theta \, dr \, d\theta, \qquad (53)$$

the radius a having been replaced by $r \sin\theta$ in the last step.

2. Then define the (total) magnetic flux of an orbiting electron in an energy eigenstate $\psi_{n\ell m}(r, \theta, \varphi)$ to be

$$\Phi_z^{(o)} = \int_0^\infty \int_0^\pi d\Phi_z^{(o)}(r, \theta, \varphi) \qquad (54)$$

$$= -\frac{(2\pi)^2 L_z}{e} \int_0^\infty \int_0^\pi |\psi_{n\ell m}(r, \theta, \varphi)|^2 \, r^2 \sin\theta \, dr \, d\theta \qquad (55)$$

$$= -\frac{2\pi}{e} L_z. \qquad (56)$$

With $L_z = m\hbar$ we get

$$\Phi_z^{(o)} = -m\Phi_e^{(o)}, \quad \Phi_e^{(o)} = h/e. \qquad (57)$$

We call $\Phi_z^{(o)}$ the *orbital magnetic flux* of the electron along the z-direction due to its orbital motion. Our result shows that the orbital magnetic flux is quantized into a multiple of $\Phi_e^{(o)} = h/e$ which may be referred to as the *electron orbital magnetic flux quantum*. For a superconductor the current is due to Cooper pairs; the corresponding *Cooper pair magnetic flux quantum* in a superconducting ring has the value $\Phi_c^{(sr)} = h/2e$ (Feynman *et al.*, 1965; Wan and Harrison, 1993).

The above derivation of the orbital magnetic flux also serves to illustrate the probabilistic nature of quantum mechanics. Because of the intrinsic probabilistic nature of quantum systems there is generally a need in quantum mechanics for an averaging process to obtain the expectation value of a physical quantity. For instance, given a wave function $\psi(x)$ the position expectation value is the average over all position values under the probability density function $|\psi(x)|^2$. Similarly our derivation of the flux is based on the following intuitive understanding:

1. The circular motion of the electron gives rise to an electric current circling the z-axis. The electron at position (r, θ, φ) executing a circule motion of radius $a = r \sin\theta$ with angular momentum L_z may be regarded as having a linear (tangential) velocity $L_z/m_e a$.
2. The electron will have only a certain probability of executing a particular circular motion since there is only a probability density $|\psi_{n\ell m}(r, \theta, \varphi)|^2$ for the electron to be at position (r, θ, φ). It follows that the probability current density circulating the z-axis at (r, θ, φ) is

$$|\psi_{n\ell m}(r, \theta, \varphi)|^2 L_z/m_e a \qquad (58)$$

which agrees with Eq. (27). This gives rise to the electric current density in Eq. (29).

3. A circulating current element would enclose a magnetic flux element. The expectation value for the magnetic flux can be obtained by summing all these flux elements, i.e., the expectation value for the magnetic flux is obtained by integrating $d\Phi_z^{(o)}(r,\theta,\varphi)$.

In analogy with a superconducting ring (Wan and Harrison, 1993) we can also introduce an *electron orbital magnetic flux operator* by

$$\widehat{\Phi}_z^{(o)} = -\frac{2\pi}{e}\widehat{L}_z \tag{59}$$

whose eigenvalues coincide with the quantized flux values, i.e.,

$$\widehat{\Phi}_z^{(o)}\psi_{n\ell m}(r,\theta,\varphi) = \left(-m\Phi_e^{(o)}\right)\psi_{n\ell m}(r,\theta,\varphi). \tag{60}$$

The orbital flux operator is related to the orbital magnetic moment operator[4] by

$$\widehat{\mu}_z^{(o)} = \frac{e^2}{4\pi m_e}\widehat{\Phi}_z^{(o)} \quad \text{or} \quad \widehat{\Phi}_z^{(o)} = \frac{4\pi m_e}{e^2}\widehat{\mu}_z^{(o)}. \tag{61}$$

Finally note that the flux operator $\widehat{\Phi}_z^{(o)}$ can be obtained by formally quantizing the classical expression in Eq. (50) with q replaced by -e. To strengthen the notion of an intrinsic magnetic flux due to orbital motion we shall give two further intuitive and semi-classical derivations.

3.3. Semi-Classical Treatment in Terms of the Bohr Atom

It is easier to appreciate the concept of an enclosed magnetic flux if one can imagine a more definite circular orbit to enclose a flux. The Bohr's model of the atom would be of help here. The electron in the Bohr's model of the hydrogen atom revolves in a certain quantized orbit of radius r_m which satisfies Bohr's angular momentum quantization rule.

$$m_e v_m r_m = m\hbar, \tag{62}$$

where v_m is the tangential velocity of the electron in that orbit and m is identifiable with the magnetic quantum number in Eq. (60). This orbit should enclose a flux Φ_m due to a current $I_m = \rho_e v_m$ where $\rho_e = -e/2\pi r_m$, arising from the circular motion of the electron. Using Eq. (40) for the self inductance (with a replaced by r_m) we get

$$\Phi_m = \mathcal{L}_e(r_m)I_m = -m\frac{h}{e} \tag{63}$$

[4] The z-component orbital magnetic moment operator is $\widehat{\mu}_z^{(o)} = \widehat{\mu}_e^{(o)} = -(e/2m_e)\widehat{L}_z$.

which agrees with Eq. (57). We can rephrase various quantities in Bohr's model in terms of the standard quantum theory of the hydrogen atom which should strengthen the quantum mechanical content of Eqs. (62) and (63). In other words we have:

1. Radius of a Bohr Orbit: The radius variable r appears in Coulomb potential energy $V(r) = -e^2/4\pi\epsilon_0 r$ in the Hamiltonian for the hydrogen atom. Or rather it is the inverse $1/r$ which appears directly. The expectation value $\langle\psi_{n\ell m}|V(r)\psi_{n\ell m}\rangle$ of $V(r)$ in the energy eigenstate $\psi_{n\ell m}$ can be shown to be equal to $V(r_n)$, where r_n is the radius of the nth Bohr orbit. In other words we have

$$\langle\psi_{n\ell m}|V(r)\psi_{n\ell m}\rangle = V(r_n) = -\frac{e^2}{4\pi\epsilon_0 r_n}, \tag{64}$$

since (Zettili, 2001)

$$\frac{1}{r_n} = \langle\psi_{n\ell m}|\frac{1}{r}\psi_{n\ell m}\rangle \tag{65}$$

We can therefore identify a Bohr radius r_n as the effective radius at which the Coulomb potential $V(r_n)$ is equal to the quantum expectation value $\langle\psi_{n\ell m}|V(r)\psi_{n\ell m}\rangle$ of the potential.

2. Velocity in a Bohr Orbit: The velocity v_m in Eq. (62) can then be regarded as the effective tangential velocity at radius r_n corresponding to the quantum expectation value $\langle\psi_{n\ell m}|\widehat{L}_z\psi_{n\ell m}\rangle$ of the z-component angular momentum operator $\widehat{L}_z$ in state $\psi_{n\ell m}$, i.e.,

$$v_m = \frac{1}{m_e r_n}\langle\psi_{n\ell m}|\widehat{L}_z\psi_{n\ell m}\rangle = \frac{1}{m_e r_n} m\hbar. \tag{66}$$

3.4. Semi-Classical Treatment in Terms of Electron's Orbital Magnetic Moment

Finally let us adopt an entirely different approach to derive the flux $\Phi_z^{(o)}$. The starting point in this approach is the generally accepted concept of magnetic moment $\mu_z^{(o)}$ associated with an orbiting electron. This magnetic moment should give rise to a magnetic field. The idea is to examine how this magnetic field may give rise to an intrinsic magnetic flux. A successful formulation of this idea will enable us to derive an intrinsic magnetic flux generated by a spin magnetic moment.

For definiteness we shall consider the energy eigenstate $\psi_{n\ell m}$ which corresponds to the magnetic moment in Eq. (37) directed along the (negative) z-direction. Let us consider this magnetic moment to be situated at the coordinate origin. Then at any point in the x–y plane a distance r from the origin, the magnetic field generated by this magnetic moment is directed along the z-axis. This field is related to $\mu_z^{(o)}$ by (Jackson, 1999)

$$B_z^{(o)} = -\frac{\mu_0}{4\pi}\frac{\mu_z^{(o)}}{r^3}. \tag{67}$$

Consider a circular disc of radius r_0 lying in the x-y plane and centered at the origin. The magnetic flux through the area of x-y plane outside the disc is

$$\Phi_{out}^{(o)}(r_0) = \int_{r_0}^{\infty}\int_0^{2\pi} B_z^{(o)} r\, d\varphi\, dr = -\frac{\mu_0}{2}\frac{\mu_z^{(o)}}{r_0}. \tag{68}$$

Since the total flux through the entire x–y plane must be zero on account of the divergence free nature of the magnetic field and the vanishing of the magnetic field at infinity we obtain the flux through the disc to be

$$\Phi_{in}^{(o)}(r_0) = \frac{\mu_0}{2}\frac{\mu_z^{(o)}}{r_0}. \tag{69}$$

Now imagine a particle of charge q going round a circular orbit of radius r_0 centered at the origin on the x–y plane. This would result in a current $I^{(o)}$ round the orbit. In order to produce a magnetic field at a point on the x-y plane at a distance $r \geq r_0$ from the origin equivalent to that produced by the magnetic moment $\mu_z^{(o)}$ at the origin we must equate the magnetic moment of the current loop to $\mu_z^{(o)}$

$$\pi r_0^2 I^{(o)} = \mu_z^{(o)}. \tag{70}$$

This current would produce a magnetic field at the center of the loop (Jackson, 1999)

$$B_0^{(o)} = \frac{\mu_0 I^{(o)}}{2r_0} = \frac{\mu_0 \mu_z^{(o)}}{2\pi r_0^3}. \tag{71}$$

The magnetic flux $\Phi_{in}^{(o)}(r_0)$ in Eq. (69) is seen to be related to B_0 by

$$\Phi_{in}^{(o)}(r_0) = \pi r_0^2 B_0^{(o)}. \tag{72}$$

In other words $\Phi_{in}^{(o)}(r_0)$ happens to be the same as that due to a constant magnetic field $B_0^{(o)}$ through the disc.

Next, let us take the flux enclosed per revolution of the particle to be $\Phi_{in}^{(o)}(r_0)$. Then the number of revolutions needed to enclose an amount of flux equal to $\Phi_z^{(o)}$ in Eq. (56) is

$$N^{(o)} = \frac{2\pi}{e}\frac{L_z}{\Phi_{in}(r_0)}. \tag{73}$$

So, starting from a magnetic moment $\mu_z^{(o)}$ at the origin aligned along the negative z-direction we can arrive at a magnetic flux through the disc with a value which is independent of the disc radius r_0.

The significance of this particular approach lies in the emphasis of the magnetic moment as the origin of the flux. In other words a quantum particle's motion which generates a magnetic moment should lead to an intrinsic magnetic flux. There should then be an intrinsic magnetic flux associated with an electron's spin. The seemingly arbitrary number $N^{(o)}$ in Eq. (73) turns out to have a definite physical origin in our study of spin in the next section.

4. SPIN MAGNETIC FLUX

The question as to whether the spin motion of an electron would give rise to an intrinsic magnetic flux is not something we can easily established since spin has no classical counterpart, i.e., we do not have a generally accepted classical picture of electron spin. However, we can try different model theories of electron spin to see if we can arrive at a common value of an intrinsic magnetic flux, i.e., an intrinsic value independent of any particular model of spin, as we did for the orbital magnetic flux. A success in such an endeavour would lend support to the suggestion on the existence of an intrinsic magnetic flux associated with spin. In what follows we shall do just that. Each model theory presented below is not meant to be taken literally as a true representation of spin. The validity of our proposal on the existence of an intrinsic magnetic flux associated with spin can only be confirmed by experiments.

4.1. Spin Magnetic Moment Approach

The starting point in this approach is the known magnetic moment associated with an electron spin. Let us denote this magnetic moment along the z-axis by $\mu_z^{(s)}$. Quantum mechanics tells us that

$$\mu_z^{(s)} = -\frac{e}{m_e} S_z, \tag{74}$$

where S_z denotes a spin value, i.e., $S_z = \pm\hbar/2$. For definiteness we shall consider a spin-up state α_z corresponding to $S_z = \hbar/2$. We can then follow the arguments of Section 3.4 to establish a similar concept of an intrinsic magnetic flux. Everything goes through as before, except with $\mu_z^{(o)}$ and L_z replaced by $\mu_z^{(s)}$ and S_z, e.g., in place of Eqs. (70), (71) and (72) we now have

$$\pi r_0^2 I^{(s)} = \mu_z^{(s)}, \quad B_0^{(s)} = \frac{\mu_0 \mu_z^{(s)}}{2\pi r_0^3}, \quad \Phi_{in}^{(s)}(r_0) = \frac{\mu_0}{2}\frac{\mu_z^{(s)}}{r_0} = \pi r_0^2 B_0^{(s)}. \tag{75}$$

If we count the flux enclosed by a similar number of revolutions as in the orbital case, i.e.,

$$N^{(s)} = \frac{2\pi}{e}\frac{S_z}{\Phi_{in}(r_0)}, \tag{76}$$

obtained from $N^{(o)}$ by replacing L_z by S_z, we will obtain an enclosed flux

$$\Phi_z^{(s)} = -\frac{2\pi}{e}S_z. \tag{77}$$

We can identify this as the intrinsic flux given rise by the spin motion. As before the radius r_0 which appears in various expressions in section 3.4 and the electron mass m_e play no part in the final expression for $\Phi_z^{(s)}$. This flux does not depend on the radius r_0. If desired one can imagine r_0 to be the classical electron radius which is of the order of 10^{-15} m (Feynman *et al.*, 1964). One can also let r_0 tend to zero. In this limit the electron may be treated as a point charge carrying a spin and an intrinsic flux of magnitude $h/2e$.

Finally let us comment on the meaning of $N^{(s)}$ here. In this model the charge going round a circle of radius r_0 would need to revolve round with a frequency.

$$f_{rev}^{(s)} = \frac{S_z}{m_e\pi r_0^2} \tag{78}$$

in order to give rise to a current $I^{(s)}$ to satisfy the first of the equations in Eqs. (75). On the other hand, under a magnetic field of magnitutde $B_0^{(s)}$ a charge has a cyclodron frequency of revolution $f_{\text{cyc}} = eB_0^{(s)}/2\pi m_e$ or a cyclotron period $T_{\text{cyc}}^{(s)} = 2\pi m_e/eB_0$. The number $N^{(s)}$ employed in the counting of flux is equal to the product of the cyclotron period $T_{\text{cyc}}^{(s)}$ time the frequency $f_{\text{rev}}^{(s)}$, i.e., $N^{(s)} = T_{\text{cyc}}^{(s)} f_{\text{rev}}^{(s)}$, the number of revolutions during a cyclotron period.

If we accept that a magnetic moment can give rise to an intrinsic magnetic flux then even electrically neutral particles like the neutron which possesses a magnetic moment can have associated with it an intrinsic magnetic flux.

4.2. Hidden Variable Approach

There have been various models of electron spin based on the idea of a spinning top (Rosen, 1951; Schulman, 1968; Schulman, 1981; Barut *et al.*, 1992). Let us begin by treating an electron as a small ball of charge spinning about its own axis. Our previous analysis should apply. Let us visualize a spherical coordinate system $(r_s, \theta_s, \varphi_s)$ with origin placed at the center of the electron. If we assume, in analogy with Eqs. (27) and (29), that the electron's spinning motion produces an electrical current density

$$j_e^{(s)}(r_s, \theta_s, \varphi_s) = -\frac{eS_z}{m_e a}\rho(r_s, \theta_s, \varphi_s), \tag{79}$$

where $\rho(r_s, \theta_s, \varphi_s)$ plays the role of $|\psi_{n\ell m}|^2(r, \theta, \varphi)$ in Eq. (27) as a probability density function for the smearing out of the electron charge. We should have a normalization condition:

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \rho(r_s, \theta_s, \varphi_s) r_s^2 \sin\theta_s \, dr_s \, d\theta_s d\varphi_s = 1. \tag{80}$$

There is no reason for $\rho(r, \theta, \varphi)$ to be the same as $|\psi_{n\ell m}|^2(r, \theta, \varphi)$, e.g., we can have $\rho(r_s, \theta_s, \varphi_s)$ going to zero outside a small radius, e.g., outside the classical electron radius. All we need is to retain cylindrical symmetry, i.e., we assume that $\rho(r_s, \theta_s, \varphi_s)$ is independent of φ_s. Following Eqs. (38) to (57) we immediately obtain the flux associated with the electron spin. For the spin-up state α_z, i.e., α_z satisfies the eigenvalue equation $\widehat{S}_z\alpha_z = +(\hbar/2)\alpha_z$, the associated magnetic flux has the value

$$\Phi_z^{(s)} = -\frac{2\pi}{e}\frac{\hbar}{2} = -\frac{1}{2}\Phi_e^{(o)} \tag{81}$$

or

$$\Phi_z^{(s)} = -\Phi_e^{(s)}, \quad \Phi_e^{(s)} = \frac{1}{2}\frac{h}{e}, \tag{82}$$

agreeing with the expression in Eq. (77). This *electron spin magnetic flux* along the z-direction can also be represented by an operator, i.e.,

$$\widehat{\Phi}_z^{(s)} = -\frac{2\pi}{e}\widehat{S}_z. \tag{83}$$

We would then get the flux as an eigenvalue of the flux operator, $\widehat{\Phi}_z^{(s)}\alpha_z = \Phi_z^{(s)}\alpha_z$.

Instead of a model electron as a spinning ball of charge in the 3-dimensional physical space $\mathbb{R}^3$ an alternative interpretation of the above formalism in terms of a hidden variable space can be offered. Let us start with the well-established hidden variable model of spin proposed by John Bell (1966, 1987). Let $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ be the usual spin operators along the *x,y* and the *z* directions respectively. Given an arbitrary spin state we can orientate our coordinate axes so that the spin state would correspond to a spin-up state along the *z*-axis, i.e., we can, without loss of generality, confine ourselves to the spin-up state α_z in our chosen coordinate system. A general spin observable has the form

$$\widehat{A}^{(s)} = a + b_x\widehat{S}_x + b_y\widehat{S}_y + b_z\widehat{S}_z, \tag{84}$$

for some real numbers a, b_x, b_y and b_z (Bell, 1966; Bell, 1987) In the traditional quantum theory of spin the expectation value of observable $\widehat{A}^{(s)}$ in state α_z is given by $\langle\alpha_z|\widehat{A}^{(s)}\alpha_z\rangle$. In Bell's hidden variable theory such an expectation value can be expressed as an integral of an appropriate numerical function $A^{(s)}(\lambda)$ of a real variable λ, known as a hidden variable, over the range $(-\frac{1}{2}, \frac{1}{2})$, i.e., there exists a real-valued function $A^{(s)}(\lambda)$ such that

$$\langle \alpha_z | \widehat{A}^{(s)} \alpha_z \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} A^{(s)}(\lambda) d\lambda. \tag{85}$$

Such a function applicable to a general spin observable $\widehat{A}^{(s)}$ can be explicitly written down. In the special case when α, b_x and b_y vanish the spin observable in Eq. (84) reduces to $\widehat{A}_z^{(s)} = b_z \widehat{S}_z$. Since

$$\widehat{A}_z^{(s)} \alpha_z = \frac{b_z \hbar}{2} \alpha_z, \tag{86}$$

we have the expectation value of observable $\widehat{A}_z^{(s)}$ in state α_z given by $\langle \alpha_z | \widehat{A}^{(s)} \alpha_z \rangle = b_z \hbar/2$. The function in Eq. (85) for $\widehat{A}_z^{(s)}$ is clearly $A_z^{(s)}(\lambda) = b_z \hbar/2$ for all $\lambda \in (-\frac{1}{2}, \frac{1}{2})$. Let us inject a geometric meaning into such a model by introducing a 3-dimensional hidden variable space Λ with rectangular coordinate axes x_λ, y_λ, z_λ similar to the usual 3-dimensional Euclidean space and its Cartesian coordinate axes (Barut and Meystre, 1984; Jackson, 1985). The corresponding spherical coordinates in Λ will be denoted by $(r_\lambda, \theta_\lambda, \varphi_\lambda)$.

Now, relate the hidden variable λ to θ_λ by

$$\lambda = -\frac{1}{2} \cos \theta_\lambda \quad \text{with} \quad \theta_\lambda \in (0, \pi) \Leftrightarrow \lambda \in \left(-\frac{1}{2}, \frac{1}{2} \right). \tag{87}$$

Substituting into Eq. (85) we get

$$\langle \alpha_z | \widehat{A}_z^{(s)} \alpha_z \rangle = \int_0^\pi A_z^{(s)}(\theta_\lambda) \frac{\sin \theta_\lambda}{2} d\theta_\lambda, \tag{88}$$

where

$$A_z^{(s)}(\theta_\lambda) = b_z \hbar/2. \tag{89}$$

The integral in Eq. (88) can be rewritten as a volume integral in the following form:

$$\langle \alpha_z | \widehat{A}_z^{(s)} \alpha_z \rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} \widetilde{A}(r_\lambda, \theta_\lambda, \varphi_\lambda) \tilde{\rho}(r_\lambda, \theta_\lambda, \varphi_\lambda) r_\lambda^2 \sin \theta_\lambda \, dr_\lambda \, d\theta_\lambda \, d\varphi_\lambda, \tag{90}$$

where $\tilde{A}(r_\lambda, \theta_\lambda, \varphi_\lambda) = b_z \hbar/2$, and

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \tilde{\rho}(r_\lambda, \theta_\lambda, \varphi_\lambda) r_\lambda^2 \sin \theta_\lambda \, dr_\lambda \, d\theta_\lambda \, d\varphi_\lambda = 1. \tag{91}$$

We can interpret this as an integral of the function $\widetilde{A}(r_\lambda, \theta_\lambda, \varphi_\lambda)$ over the hidden variable space Λ weighted by the probability density function $\widetilde{\rho}(r_\lambda, \theta_\lambda, \varphi_\lambda)$ in the hidden variable space Λ. When the function $\widetilde{A}$ and the probability density function

$\widetilde{\rho}(r_\lambda, \theta_\lambda, \varphi_\lambda)$ are independent of φ_λ the integrals in Eqs. (90) and (91) reduce to

$$\langle \alpha_z | \widehat{A}_z^{(s)} \alpha_z \rangle = 2\pi \int_0^\infty \int_0^\pi \widetilde{A}(r_\lambda, \theta_\lambda) \widetilde{\rho}(r_\lambda, \theta_\lambda) r_\lambda^2 \sin\theta_\lambda \, dr_\lambda \, d\theta_\lambda, \tag{92}$$

and

$$\int_0^\infty \int_0^\pi \widetilde{\rho}(r_\lambda, \theta_\lambda) r_\lambda^2 \sin\theta_\lambda \, dr_\lambda \, d\theta_\lambda = \frac{1}{2\pi}. \tag{93}$$

Next let us imagine an electron in the spin-up state as a ball spinning round about the z_λ-axis with angular momentum S_{z_λ} in the hidden variable space Λ. Following Eqs. (27) and (29) we can imagine a corresponding linear velocity of $S_{z_\lambda}/m_e a_\lambda$ at position $(r_\lambda, \theta_\lambda, \varphi_\lambda)$ for the spinning motion in the hidden variable space with radius $a_\lambda = r_\lambda \sin\theta_\lambda$ and that the linear motion gives rise to an electric current density $-eS_{z_\lambda}/m_e a_\lambda$. Following previous arguments and the assumption that $S_{z_\lambda} = \hbar/2$ we would obtain the flux $\Phi_z^{(s)}$ in Eq. (82). This result is perhaps not surprising as we have employed essentially the same mathematical analysis. This approach which relates to the well-known Bell's hidden variable theory formulated in a 3-dimensional hidden variable space may be more appealing to people who do not wish to visualize an electron as a spinning ball of charge in the physical space.

5. CONCLUDING REMARKS

An orbiting electron in an atom encloses a magnetic flux due to its orbital motion. In addition we would argue that an electron possesses an intrinsic magnetic flux due to its spin, i.e., we can imagine an electron as a kind of flux tube. In addition to its intrinsic flux an electron executing a circular motion due to an external magnetic field also encloses an external magnetic flux given by Eq. (1). The present result can be associated with the recent investigation of excitonic transport in semiconductor nano-structure which has been attracting an increasing interest (Miller *et al.*, 1999). On account of the similarities, excitons in semiconductors may be regarded as "atoms" each consisting of an electron and a hole that are bound together by Coulombic attractions. Such excitons resemble those two-level systems important for optoelectronic devices.

We have now three basic magnetic flux quanta:

1. The magnetic flux quantum due to the electron's orbital motion in an atom: the electron orbital magnetic flux quantum $\Phi_e^{(o)} = h/e$.
2. The magnetic flux quantum due to the electron's spin motion: the electron spin magnetic flux quantum $\Phi_e^{(s)} = h/2e$.
3. The magnetic flux quantum due a supercurrent in a superconducting ring; the Cooper pair magnetic flux quantum $\Phi_c^{(sr)} = h/2e$.

The Cooper pair flux quantum $\Phi_c^{(sr)} = h/2e$ has a magnitude of 2.07×10^{-7} gauss cm^2. This value was measured by Deaver and Fairbank back in 1961 (Deaver and Fairbank, 1961; Fujita and Godoy, 1996; Willaims, 1970; Gallop, 1091). The magnitude of the electron orbital flux quantum $\Phi_e^{(o)}$ is twice that of $\Phi_c^{(sr)}$. An optical transition between different energy levels of an electron in an atom should incur a change of flux, e.g., a transition from an excited state of magnetic quantum number m to the ground state of zero magnetic quantum number means a change of magnetic flux by $m\Phi_e^{(o)}$ which is many times bigger than $\Phi_c^{(sr)}$. A spin-flip of a free electron should also involve a change of flux. It would be interesting to devise an experiment to test Faraday's and Lenz's laws arising from the change of flux due to these transitions, and to explore the applications of these intrinsic quantities.

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